

Invisibility in billiards is impossible in an infinite number of directions

Alexander Plakhov* Vera Roshchina†

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Abstract

We show that the maximum number of directions of invisibility in a planar billiard defined in the exterior of a piecewise smooth body is at most finite.

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1 Introduction

The purpose of this work is to prove that billiard invisibility is impossible in an infinite number of directions in a class of bounded two-dimensional bodies with a piecewise smooth boundary. Invisibility in a given direction means that almost all billiard particles that initially move in this direction and hit the body are eventually redirected (after several reflections) to the same trajectory, so the initial and final infinite intervals of the particle's trajectory lie on the same straight line.

The research on cloaking and invisibility is flourishing, and the range of models were designed for a variety of settings including acoustic and electromagnetic cloaking and the design of cloaking metamaterials [3, 5–8].

Our focus is on the billiard model of invisibility, where the concealed object is covered with a mirror surface that reflects the light rays back to their original trajectory, hence cloaking the object.

We showed earlier that it is impossible to construct a body invisible in *all* directions [11], and it was also shown recently that it is possible to construct planar bodies invisible in any finite number of directions [9].

This work serves to lower the known upper bound on the number of directions of invisibility for piecewise smooth bodies: in the two-dimensional case it is now reduced from ‘less than all’ to ‘at most finite number’ of directions. We note however that the known constructions of bodies invisible in several directions (including the aforementioned work [9] and the constructions in [12] and [13]) are not piecewise smooth.

*Center for R&D in Mathematics and Applications, Department of Mathematics, University of Aveiro, Portugal and Institute for Information Transmission Problems, Moscow, Russia, plakhov@ua.pt

†RMIT University, Federation University Australia and the University of New South Wales, Australia, vera.roshchina@rmit.edu.au

In a somewhat different development Burdzy and Kulczycki [4] showed that it is possible to construct a body that is almost invisible in almost all directions with a given arbitrarily small accuracy. The body they constructed is the finite union of disjoint line segments contained in the unit circle, and its projection on most directions is close to the corresponding projection of the circle. Here ‘most’ and ‘close’ mean up to a set of arbitrarily small measure. We also note a light-hearted take on invisibility [15] where overlapping projections are used to conceal part of the perceived mirror reflection of the actual object.

Note also several constructions of invisible bodies provided in the papers [1] (invisibility in one direction), [11] (invisibility in two directions), [12] (invisibility in two and three directions), [10] (invisibility from one point), and [13] (invisibility from two points). Some examples of invisible bodies are given in figures 1, 2, and 3.

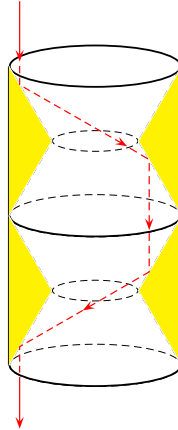


Figure 1: A body invisible in the vertical direction.

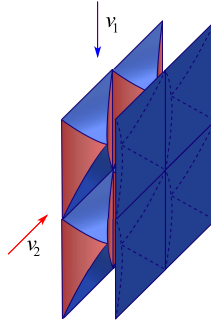


Figure 2: A body invisible in the directions v_1 and v_2 .

We also mention the following somewhat related results. In the paper [2], non-flat Riemannian metrics in \mathbb{R}^n are constructed which are Euclidean outside a compact set and have $n(n+1)/2$ directions of invisibility. In the paper [14], transformations of parallel bundles of light rays generated by specular reflections are studied.

Our main result is as follows.

Theorem 1. *There are no bounded piecewise smooth bodies invisible in infinitely many directions in \mathbb{R}^2 .*

In Section 2 we go over basic definitions that allow for rigorous interpretation of Theorem 1, and provide the proof in Section 3.

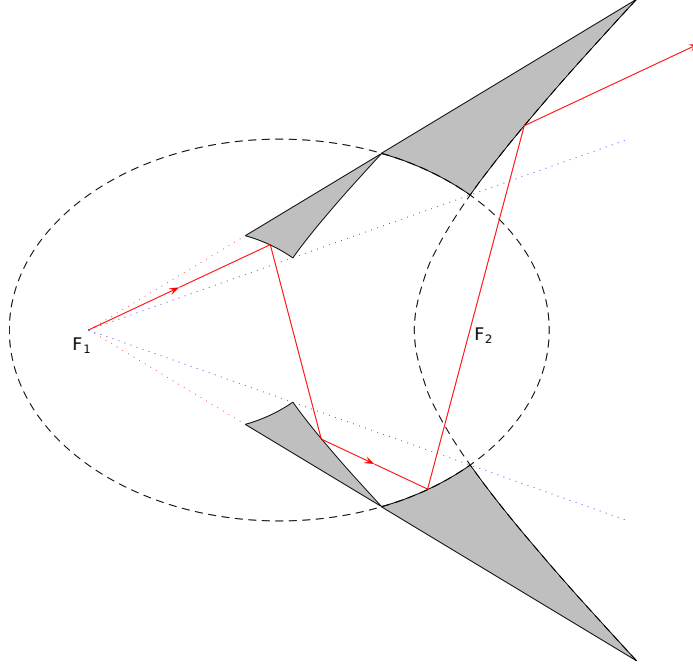


Figure 3: A 2-dimensional body invisible from the point F_1 . By rotating it about the line F_1F_2 one obtains a 3-dimensional invisible body.

2 Definitions and notation

Definition 1. A set $\gamma \subset \mathbb{R}^2$ is called a *piecewise smooth 1D set*, if it is the union of a finite number of curves,

$$\gamma = \cup_{i=1}^m \gamma_i([a_i, b_i]),$$

satisfying the following conditions: each curve is smooth, has a non self-intersecting interior, and the interiors of different curves are disjoint. That is, each function $\gamma_i : [a_i, b_i] \rightarrow \mathbb{R}^2$ is of class C^1 (this implies that there exist the lateral derivatives of γ_i at a_i and b_i); $|\gamma_i'(t)| \neq 0$ for all $t \in [a_i, b_i]$; $\gamma_i(t_1) = \gamma_i(t_2)$ for $t_1, t_2 \in [a_i, b_i]$ implies either $t_1 = t_2$, or $\{t_1, t_2\} = \{a_i, b_i\}$; and for $i \neq j$, $\gamma_i([a_i, b_i]) \cap \gamma_j([a_j, b_j]) = \emptyset$.

The endpoints of the curves $\gamma_i(a_i)$, $\gamma_i(b_i)$, $i = 1, \dots, m$ are called *singular points* of γ , and the rest of the points of γ are called *regular*.

Definition 2. A compact set $B \subset \mathbb{R}^2$, whose boundary ∂B is a piecewise smooth 1D set, is called a *body*. *Regular* and *singular* points of the body's boundary are determined in agreement with Definition 1.

An example of a body is shown in Fig. 4.

Remark 1. Note that according to Definition 2, a body is *not necessarily connected*.

We consider the billiard in $\mathbb{R}^2 \setminus B$, where B is a body.

Definition 3. A billiard motion $x(t)$, $x'(t)$ is called *regular*, if it is defined for all $t \in \mathbb{R}$ and has a finite number of reflections at regular points of ∂B .

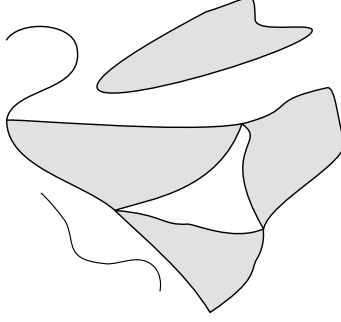


Figure 4: A body.

According to this definition, the function $x(t)$ describing a regular billiard motion is piecewise linear, and its graph has a finite number of linear segments, with the initial and final segments being unbounded:

$$x(t) = x + vt, \quad t \leq t_i; \quad x(t) = x^+ + v^+t, \quad t \geq t_f.$$

Here t_i and t_f indicate the instants of the first and the final reflection of the billiard particle. Besides, the velocity $x'(t)$ of the particle is a piecewise constant function taking values in S^1 .

The values x, v are called *the initial data* and x^+, v^+ *the final data* of the motion. The final data are functions of the initial ones, $x^+ = x^+(x, v)$, $v^+ = v^+(x, v)$. These functions are continuous and defined on an open subset of $\mathbb{R}^2 \times S^1$. Each regular billiard motion is uniquely defined by its initial data (and also by its final data).

Remark 2. If $x_1 - x_2$ is parallel to v then the two motions with the initial data x_1, v and x_2, v can be obtained one from the other by a shift along t (and therefore the corresponding trajectories $\{x_1(t), t \in \mathbb{R}\}$ and $\{x_2(t), t \in \mathbb{R}\}$ coincide). Therefore the billiard motion is well defined even if x is contained in B (see Fig. 5). In this case the particle initially moves along the ‘negative’ half-line with the direction vector v until a collision with B .

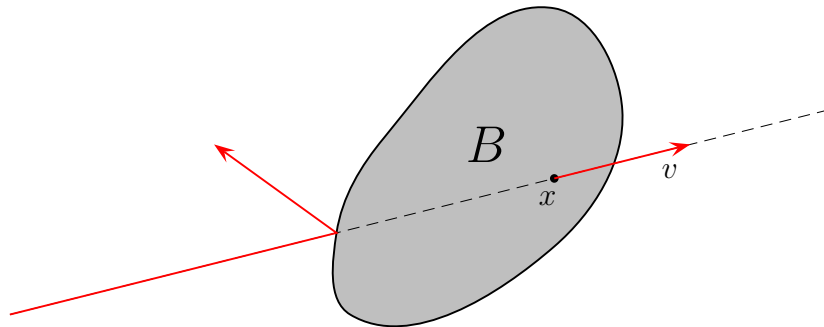


Figure 5: The motion with the initial data x, v .

Definition 4. A regular billiard motion is *not perturbed* by the body B (or just *unperturbed*), if for its initial x, v and final x^+, v^+ data we have $v^+ = v$ and $x^+ - x$ is parallel to v .

Remark 3. It follows from this definition that the (unbounded) initial and final linear segments of the trajectory corresponding to an unperturbed motion lie on a single straight line.

We assume that $v \in S^1$, i.e. the billiard particles move with unit velocity.

Definition 5. A body B is said to be *invisible in the direction v* , if for almost all $x \in \mathbb{R}^2$ the billiard motion with the initial data x, v is regular and not perturbed by B . The vector v is called *a direction of invisibility* for B .

Remark 4. If B is invisible in a direction v , then it is also invisible in the direction $-v$.

Remark 5. For all invisible bodies we know, each billiard motion in a direction of invisibility is either unperturbed, or hits the body's boundary at a singular point.

Introduce some notation. The vector obtained by counterclockwise rotation of v by $\pi/2$ is denoted by v^\perp . Thus we have $(v^\perp)^\perp = -v$, $\langle u, v \rangle = \langle u^\perp, v^\perp \rangle$, and $\langle u, v^\perp \rangle = -\langle u^\perp, v \rangle$. Here and in what follows $\langle \cdot, \cdot \rangle$ denotes the scalar product in \mathbb{R}^2 .

Let $\xi \in \partial B$ be a regular point; then $n(\xi)$ denotes a unit normal to ∂B at ξ . (We choose the unit normals so as for each i , $n(\gamma_i(t))$ depends continuously on $t \in [a_i, b_i]$.) $\text{Conv } B$ denotes the convex hull of B . The set $\partial B \cap \partial(\text{Conv } B)$ is called the *convex part* of the boundary of B .

3 Proof of the main result

To prove Theorem 1 we need several technical results.

Lemma 1. *If there exists a regular billiard motion with initial data x, v such that $v^+(x, v) \neq v$, then B is not invisible in the direction v .*

Proof. Since the function v^+ is continuous, for \check{x} in a small neighborhood of x we have $v^+(\check{x}, v) \neq v$. Therefore B is not invisible in the direction v . \square

Lemma 2. *Let B be invisible in two linearly independent directions v_1 and v_2 . Then each point of $\partial B \cap \partial(\text{Conv } B)$ is a singular point of ∂B .*

Proof. Suppose that $\xi \in \partial B \cap \partial(\text{Conv } B)$ is a regular point of ∂B . Then it is also a regular point of $\partial(\text{Conv } B)$. At least one of the values $\langle v_1, n(\xi) \rangle$, $\langle v_2, n(\xi) \rangle$ is nonzero. Without loss of generality assume that $\langle v_1, n(\xi) \rangle \neq 0$. Replacing if necessary v_1 with $-v_1$, we can ensure that the half-line $\xi + v_1 t$, $t < 0$ lies entirely outside $\text{Conv } B$.

The billiard motion in $\mathbb{R}^2 \setminus \partial(\text{Conv } B)$ with the initial data ξ, v_1 is regular and is described by the function

$$x(t) = \begin{cases} \xi + v_1 t, & \text{if } t \leq 0 \\ \xi + v_1^+ t, & \text{if } t \geq 0 \end{cases},$$

where $v_1^+ = v_1 - \langle v_1, n(\xi) \rangle n(\xi)$ is defined according to the law of elastic reflection (see Fig. 6). Obviously, $v_1^+ \neq v_1$. Note that the same motion is also a regular motion in $\mathbb{R}^2 \setminus B$, and so by Lemma 1, B is not invisible in the direction v_1 . \square

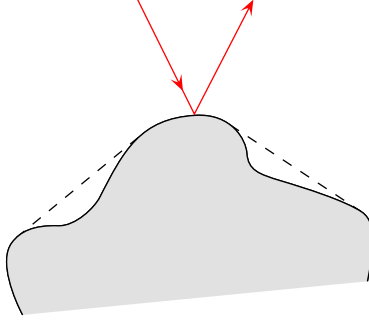


Figure 6: Reflection from the convex part of the body's boundary.

Corollary 1. *Under the assumptions of Lemma 2, $\text{Conv } B$ is a polygon.*

Proof. Indeed, all the extreme points of $\text{Conv } B$ belong to $\partial B \cap \partial(\text{Conv } B)$; therefore they are singular points of ∂B . It follows that the set of extreme points is finite, therefore $\text{Conv } B$ is a polygon. \square

Definition 6. Let

$$a_n = \sup_{x \in B} \langle x, n \rangle.$$

The line $\langle x, n \rangle = a_n$ is called the n -supporting line for the body B . The set $B \cap \{x : \langle x, n \rangle = a_n\}$ is called the n -maximal set of B , and points of this set are called n -maximal points of B .

Lemma 3. *Under the assumptions of Lemma 2, the v_1^\perp -maximal set of B contains at least two points.*

Proof. For the sake of simplicity write v in place of v_1 , omitting the subscript. Since B is compact, the v^\perp -maximal set is not empty. It remains to prove that it is not a singleton. Assume the contrary, that is,

$$B \cap \{x : \langle x, v^\perp \rangle = a_{v^\perp}\} = \{\xi\}.$$

Consider the curves γ_i comprising ∂B that contain ξ (see Fig. 7). Clearly, ξ is an endpoint of each such curve. Assume without loss of generality that ξ is the second endpoint of each curve, $\xi = \gamma_i(b_i)$, and therefore

$$\langle \gamma'_i(b_i), v^\perp \rangle \geq 0.$$

(Otherwise we just reparameterize the curve by $\tilde{\gamma}_i(t) = \gamma_i(-t)$, $t \in [-b_i, -a_i]$.)

Recall that by Corollary 1, $\text{Conv } B$ is a polygon, therefore B is contained in an angle with the vertex at ξ , which is (except for the vertex) contained in the half-plane $\langle x, v^\perp \rangle < a_{v^\perp}$. This fact allows one to sharpen the previous inequality,

$$\langle \gamma'_i(b_i), v^\perp \rangle > 0.$$

This also implies that in the coordinate system with the first axis parallel to v^\perp and the second axis parallel to $-v$, the curves γ_i containing ξ can be locally (in a small strip $a_{v^\perp} - \varepsilon < \langle x, v^\perp \rangle < a_{v^\perp}$) represented by graphs of functions. Moreover, these functions

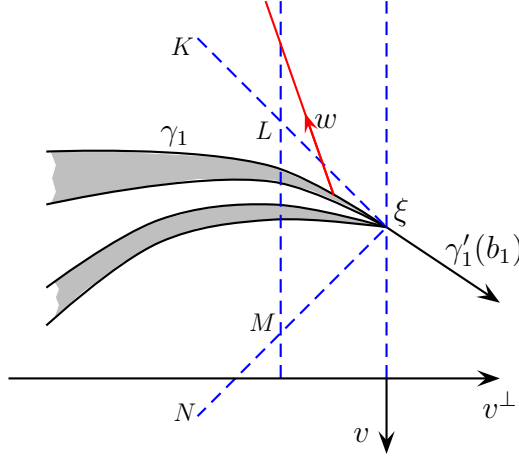


Figure 7: The v^\perp -maximal set is a singleton.

form a finite monotone sequence. Let the graph of the largest function represent the curve γ_1 .

One can choose $\varepsilon > 0$ in such a way that the aforementioned angle (and therefore B) is contained in the angle

$$A_\varepsilon = \left\{ x : \left\langle \frac{x - \xi}{|x - \xi|}, v^\perp \right\rangle \leq -2\varepsilon \right\}.$$

(the angle $\angle K\xi N$ in Fig. 7).

We have

$$\text{either (i) } \max_i \langle \gamma'_i(b_i), v \rangle \geq 0, \quad \text{or (ii) } \min_i \langle \gamma'_i(b_i), v \rangle \leq 0$$

(with the maximum and minimum taken over all i such that $\gamma_i(b_i) = \xi$). Assume that (i) holds (the case (ii) is considered in a similar way). By our convention, the maximum is achieved for γ_1 , therefore

$$\langle \gamma'_1(b_1), v \rangle = \max_i \langle \gamma'_i(b_i), v \rangle \geq 0.$$

Then for all x with $a_{v^\perp} - \langle x, v^\perp \rangle$ positive and sufficiently small, the motion with the initial data x, v has (within the strip) a single reflection at a point $x_1 = x_1(x)$ of the curve γ_1 . Besides, the velocity $v^+(x)$ after the reflection is close to

$$v^+ = v - 2\langle v, \gamma'_1(b_1)^\perp \rangle \gamma'_1(b_1)^\perp,$$

which means that $|v^+(x) - v^+| < \varepsilon$.

One easily sees that $\langle v^+, v^\perp \rangle \geq 0$; indeed,

$$\langle v^+, v^\perp \rangle = -2\langle v, \gamma'_1(b_1)^\perp \rangle \langle \gamma'_1(b_1)^\perp, v^\perp \rangle = 2\langle v^\perp, \gamma'_1(b_1) \rangle \langle \gamma'_1(b_1), v \rangle \geq 0.$$

It follows that $\langle v^+(x), v^\perp \rangle > -\varepsilon$.

If $a_{v^\perp} - \langle x, v^\perp \rangle$ is positive and sufficiently small then each ray $x_1 + wt$, $t \geq 0$ with the vertex at $x_1 = x_1(x)$ and $|w| = 1$, $\langle w, v^\perp \rangle > -\varepsilon$ is entirely contained in the set

$$V_\varepsilon = \left\{ x : \langle x, v^\perp \rangle > a_{v^\perp} - \varepsilon, \left\langle \frac{x - \xi}{|x - \xi|}, v^\perp \right\rangle > -2\varepsilon \right\}.$$

(the set bounded by and to the right of the broken line $KLMN$ in Fig. 7). This implies that each particle with the corresponding initial data x, v makes a single reflection from γ_1 and then moves freely forever. Indeed, the further motion is in V_ε , which is the union of the strip $a_{v^\perp} - \varepsilon < \langle x, v^\perp \rangle < a_{v^\perp}$ and the set $\mathbb{R}^2 \setminus A_\varepsilon$ (exterior of the angle). There are no further reflections inside the strip, and no point of reflection can belong to $\mathbb{R}^2 \setminus A_\varepsilon$ (since $B \subset A_\varepsilon$).

It remains to note that $\langle v, \gamma'_1(b_1)^\perp \rangle < 0$ and therefore $v^+(x) \neq v$. We come to a contradiction with Lemma 1, which states that B is not invisible in the direction v . \square

The following corollary of Lemma 3 is obvious.

Corollary 2. *Under the assumptions of Lemma 2, v_1 is parallel to a side of the polygon $\text{Conv } B$.*

We are now ready to prove Theorem 1. Let a body B be invisible in several (at least two) directions. By Corollary 1, $\text{Conv } B$ is an m -gon, $m \geq 3$. By Corollary 2, each direction of invisibility is parallel to a side of this m -gon. Therefore there are at most m directions of invisibility. The theorem is proved.

Acknowledgements

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